Moment of inertia of different bodies

Aim: Study moment of inertia of different bodies

Objectives of the experiment:
1. Measuring the period of oscillation of a thin transverse rod with weights on a torsion axle as a function of the distance of the weights from the torsion axle.
2. Confirming the proportionality between the moment of inertia of the weights and the square of the distance.
3. Determining the restoring torque of the torsion axle.

Apparatus: Torsion axle, spring, rod with weights, different shape objects, spring, stand base, stop clock

Theory: The moment of inertia is a measure of the inertia that a body exhibits when a torque acts on it causing a change of its rotational motion. In rotational oscillations, the period of oscillation 'T' is directly proportional to the square root of moment of inertia 'J' of the oscillating system.

More specifically: $T = 2\pi\sqrt{\frac{J}{D}}$…………(I)   D' is restoring torque

The moment of inertia of a point like mass 'm' moving on a circular path with radius 'r' is $J_1 = mr^2$……(II).

The moment of inertia of two equal masses 'm' that is rigidly connected and has the same distance 'r' from the axis of rotation is
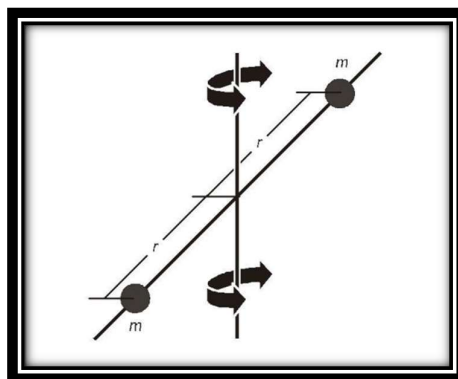$J_2 = 2mr^2$ ……(III).

Fig. 1 schematic illustration

In both cases, the moment of inertia is proportional to the square of the distance 'r'.

In the experiment, the rigid connection between the two masses is established by means of a thin rod whose middle is fixed to the torsion axle (see Fig. 1). After deflection from the equilibrium position, the system oscillates with the period of oscillation 'T'. From Eq. (I) it follows that

$$J = D \left(\frac{T}{2\pi}\right)^2 \ldots\ldots\ldots\ldots\ldots\ldots.(IV)$$

However, the moment of inertia is composed of the moment of inertia' J_2' of the two weights and the moment of inertia J_0 of the rod:

$$J = 2mr^2 + J_0 \quad \ldots..(V)$$

Thus, $D\frac{T^2}{4\pi} = 2mr^2 + J_0$

Therefore, the period of oscillation 'T_0' of the rod without weights is measured in another measurement, which leads to

$$D\left(\frac{T}{2\pi}\right)^2 = 2mr^2 + D\left(\frac{T_0}{2\pi}\right)^2$$

or

$$T^2 = \frac{8m\pi^2}{D}r^2 + T_0{}^2 \ldots\ldots\ldots.(VI)$$

Thus a linear relation between the square of the period of oscillation 'T' and the square of the distance' r' is obtained. From the slope of the straight line,

$$a = \frac{8m\pi^2}{D}\ldots\ldots\ldots\ldots\ldots\ldots.. (VII)$$

The restoring torque 'D' can be calculated if the mass 'm' is known.

Setup and carrying out the experiment

The experimental setup is illustrated in Fig. 2.

- Fix the middle of the transverse rod to the torsion axle with the screw and arrange the weights symmetrically at a distance of 30 cm from the torsion axle.

- Mark the equilibrium position on the table.

- Rotate the transverse rod to the clock-wise in between 10°- 180° and release it. Time period is independent of the amplitude. Test this before starting the experiment by measuring the time period with different amplitudes.

- Start the time measurement as soon as the transverse rod passes through the equilibrium position and stop the measurement after five oscillations.

- Repeat the measurement five times, alternately deflecting the rod to the clock-wise and anticlock-wise.

- Calculate the period of oscillation 'T' from the mean value of the five measured values.

- One after another reduce the distance to 25 cm, 20 cm, 15 cm, 10 cm and 5 cm, each time repeating the measurement.

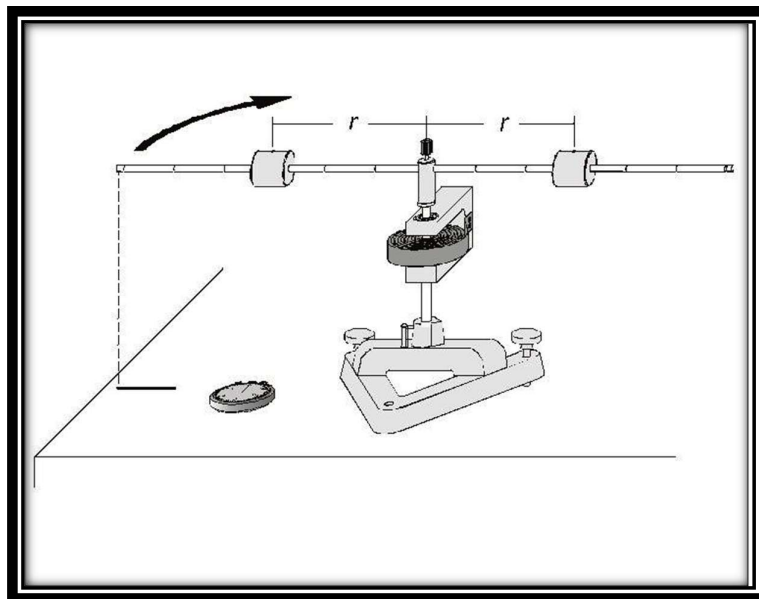- Remove the weights, and repeat the measurement.

Fig. 2. Experiment setup to determine the period of oscillation.

Observations:

Table-1: Measurement duration of five oscillations for various distances 'r' between the torsion axle and the weights and mean value of the period of oscillation 'T' calculated from the measured value.

No. of obs.	r (cm)	r^2 (cm^2)	Time period for 5 oscillations (5T in seconds)					Avg. of 5T (s)	T (s)	T^2 (s^2)
	Without weights									

Calculations

From Table 1, we obtain T_0 = ………s and therefore T_0^2 = ……s^2 and the squares of the oscillation periods of the system with weights.

Plot a graph between T^2 and r^2. Linear fit the line to determine slope and intercept and error in slope and error in intercept.

As the mass 'm' is known, the restoring torque 'D' can be calculated from the slope using equation (VII).

D = ……………. Newton-meter

Determine the error in 'D' using by propagating the error in mass 'm' and error in slope 'a'.

ΔD =…………….Newton-meter

Objectives of the experiment (Part 2):

➤ Determining the moments of inertia of rotationally symmetric bodies from their period of oscillation on a torsion axle.
➤ Comparing the periods of oscillation of two bodies having different masses, but the same moment of inertia.
➤ Comparing the periods of oscillation of hollow bodies and solid bodies having the same mass and the same dimensions.
➤ Comparing the periods of oscillation of two bodies having the same mass and the same body shape, but different dimensions.

Theory: The moment of inertia is a measure of the resistance of a body against a change of its rotational motion and it depends on the distribution of its mass relative to the axis of rotation. For a calculation of the moment of inertia 'J', the body is subdivided into sufficiently small mass elements 'Δm_i' with distances 'r_i' from the axis of rotation and a sum is taken over all mass elements:

$$J = \sum_i \Delta m_i r_i^2 \ldots\ldots\ldots\ldots(1)$$

For bodies with a continuous mass distribution, the sum can be converted into an integral. If, in addition, the mass distribution is homogeneous, the integral reads

$$J = \frac{M}{V} \int_V r^2 \, dV \ldots\ldots\ldots \quad (2)$$

The calculation of the integral is simplified when rotationally symmetric bodies are considered which rotate around their axis of symmetry. The simplest case is that of a hollow cylinder with radius 'R'. As all mass elements have the distance 'R' from the axis of rotation, the moment of inertia of the hollow cylinder (J_{HC}) is

$$J_{HC} = MR^2 \ldots\ldots\ldots\ldots (3)$$

In the case of a solid cylinder (J_{SC}) with equal mass M and equal radius R, Eq. (2) leads to the formula

$$J_{SC} = \frac{M}{V} \int_0^R r^2 (2\pi r) H dr \quad where \quad V = \pi R^2 H$$

and the result is $J_{SC} = \frac{1}{2} MR^2 \ldots\ldots\ldots\ldots (4)$

That means, the moment of inertia of a solid cylinder is smaller than that of the hollow cylinder as the distances of the mass elements from the axis of rotation are between Zero (0) and 'R'.
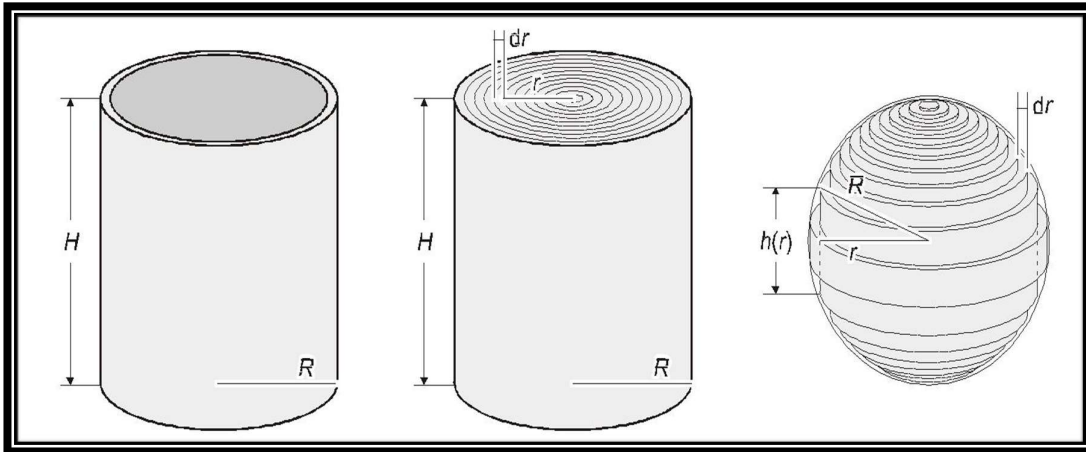

Fig. 3: Calculation of the moments of inertia of a hollow cylinder, a solid cylinder and a sphere

An even smaller value is expected for the moment of inertia of a solid sphere with radius 'R' (see Fig. 1). In this case, Eq. (2) leads to the formula

$$J_{SS} = \frac{M}{V} \int_0^R r^2 2\pi r (2\sqrt{R^2 - r^2}) dr \qquad where \quad V = \frac{4\pi}{3} R^3$$

and the result is $J_{SS} = \frac{2}{5} MR^2$ ……… (5)

Thus, apart from the mass 'M' and the radius 'R' of the bodies under consideration a dimensionless factor enters the calculation of the moment of inertia, which depends on the shape of the respective body.

The moment of inertia is determined from the period of oscillation of a torsion axle, on which the test body is fixed and which is connected elastically to the stand via a helical spring. The system is excited to perform harmonic oscillations. If the restoring torque 'D' is known, the moment of inertia of the test body is calculated from the period of oscillation T according to

$$J = D \left(\frac{T}{2\pi}\right)^2 \ldots\ldots\ldots\ldots (6)$$

Setup and carrying out the experiment:

- Put the sphere on the torsion axle, and mark the equilibrium position on the table.
- Rotate the sphere to the clock-wise in between 10-180° and release it.
- Start the time measurement as soon as the sphere passes through the equilibrium position and stop the measurement after five oscillations (5T).
- Calculate the period of oscillation 'T'.
- Replace the sphere and repeat the measurement.
- After the measurement of time period for the sphere, repeat the measurement with different bodies, namely disk, solid cylinder and hollow cylinder. For the solid cylinder and hollow cylinder, use a supporting plate to hold them.
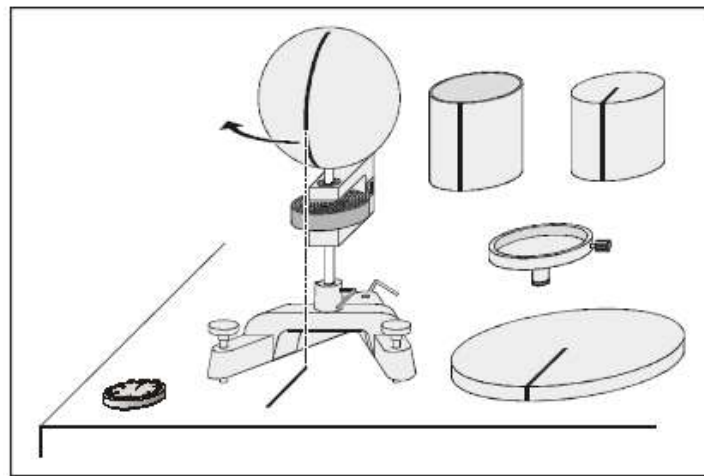- Finally carry out the measurement with the empty supporting plate.

Fig.4: Setup for measuring time period

Data:

Table 2: List of the bodies under consideration and the measured oscillation periods

Body	M(gm)	2R(cm)	5T(sec)	T(sec)
Solid sphere				
Flat solid cylinder (disk)				
Long solid cylinder				
Hollow cylinder				
Empty supporting plate				

Analysis:

a) Qualitative comparison:

- Bodies having different masses, but the same moment of inertia:
  The sphere and the flat solid cylinder (disk) have different shapes and different masses. They oscillate with approximately the same period, i.e. they have the same moment of inertia.
- Hollow body and solid body:
  The hollow cylinder and the solid cylinder have approximately the same mass and the same diameter. The periods of oscillation are clearly different, i.e. they have different moments of inertia.
- Bodies having the same mass and the same shape but different dimensions:
  The flat solid cylinder (disk) and the long solid cylinder have approximately the same mass, but different diameters. They oscillate at clearly different periods, i.e. the moments of inertia are different.

b) Quantitative comparison:

Using Eq. (6), the moments of inertia 'J' can be calculated for each case using the periods 'T' listed in Table 2.

Calculate dimensionless factors of Eqs. (3), (4) and (5) and compare with the values obtained from the measured data.

Table 3:

Body	MR^2 (gm.cm^2)	Dimensionless factor (J/MR^2)	$J = D\left(\dfrac{T}{2\pi}\right)^2$	Experiment (J/MR^2)	Comparison
Solid sphere					
disk					
Solid cylinder					
Hallow cylinder					

Note: Since supporting plate is used to hold the cylinders while measuring the time period of cylinders, the moment of inertia of the supporting plate needs to be subtracted from the moment of inertia of the cylinders to get only the moment of inertia of the cylinders.

Calculations:

For solid sphere, disk, solid cylinder and hallow cylinder, calculate the error in moment of inertia by propagating error in 'D' and 'T'.